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Inflationary Fallacies

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It is proven that the chaotic inflationary scenario is not realistic and in many essential points is false.

1. Inflationary cosmological scenarios are very popular, and many works, reports, and surveys on inflation have been published; in fact, it has become natural to refer to an inflationary paradigm [some reviews even carry this as their title (Turner, 1986)]. In this report I prove that the inflationary scenarios are not realistic and in many essential points are false.[2]

Following the pioneering work of Guth (1981), various scenarios of an inflationary universe were proposed [for recent reviews see Linde (1984*a*) and Turner (1986)]. The aim of these inflationary scenarios is the solution [without using the anthropic principle—on the use of this principle see Sakharov (1984), for example] of many cosmological problems and the explanation of many properties of the observable (homogeneous, flat, monopole-free) universe. With the natural assumption of the uniqueness of the unified universe,[3] the explanation of the observable properties of the unique (unified) universe on the basis of a unique (exotic) mechanism or on the basis of unique (exotic) initial conditions hardly can be considered as a sufficiently meaningful scientific explanation. But the proposal in Albreicht and Steinhardt (1982) and Linde (1983*a*) of a "new scenario"

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[2]This report is based on my previous work (Khalfin, 1984, 1985, 1986, 1987*a*, 1988) and also gives some new results.

[3]This is factual, not an assumption, but the tautological definition of the unique unified universe.

based on phase transitions like the Coleman–Weinberg model with fine tuning looks like such an exotic "nonscientific" explanation. One of the author's of the "new scenario" now states[4] that it "was originated with the impression that the inflation (durationally exponential inflation) of the universe is a sufficiently exotic phenomenon ... which can be realized only in very limited sets of theories" (Linde, 1983*a*). In addition, Hawking (1988) states, "On my personal point of view the new inflationary model is now dead as scientific theory, although a lot of people do not seem to have heard of its demise and are still writing papers as if it were viable." Besides intrinsic obstacles of the "new scenario," which were discussed in Linde (1984*a*), I point out that all inflation scenarios, which are based on phase transitions, use for an estimation of the duration period of the inflation (the kinetics of the phase transition) a Euclidean approach, in particular the Langer–Polyakov–Coleman instanton method. However, I recently proved (Khalfin, 1987*b*) that this very popular method is incorrect. I do not go into the details of this proof (Khalfin, 1987*b*), but only point out here that the reason for the incorrectness of the Langer–Polyakov–Coleman method corresponds to the fact that the nonexponential (Khalfin, 1957, 1958, 1960) terms in the physical (Minkowski) amplitude of the decay are essential only for very big times, but for the Euclidean amplitude of the decay, which is evaluated in the Langer–Polyakov–Coleman method, the nonexponential terms are essential (bigger than exponential ones) for finite times. The problem of the evaluation of a duration period for inflationary (the kinetics of the phase transitions) scenarios, which are based on phase transitions, will be discussed in a separate work.

Linde (1983*a*,*b*) proposed the chaotic inflationary scenario: "Thus, the inflation of the universe is not an exotic phenomenon, which is only possible in some special Coleman–Weinberg models, but it is a natural consequence of chaotic initial conditions in the early universe, which are realized in a wide class of elementary particles theories" (Linde, 1983*a*). Besides solutions of many cosmological problems, as in previous inflationary scenarios (Guth, 1981; Linde, 1982; Albreicht and Steinhardt, 1982), the chaotic inflationary scenario, as stated by its author, reduces to a new and cardinal deduction on the formation of an infinitely large number of "mini"-universes which can have some properties different from properties of "our" universe. Thus, the chaotic inflationary scenario predicts, the essentially "macro" inhomogeneity of the unique (unified) universe. Linde (1984*a*) stated that the realization of the "new" scenario inside the relict inflation in supergravity is possible only in the chaotic scenario. Linde (1984*a*) also stated that the quantum origin of the universe [see, for example, Linde (1984*b*)] can be

[4] I expressed such an opinion to A. D. Linde immediately after his first report (Linde, 1981) on the "new scenario."

realized in the chaotic scenario, too. Recent work (Linde, 1986; Goncharov and Linde, 1987) on the chaotic scenario has proposed the idea of an eternally existing, self-reproducing, chaotic inflationary universe.

I have shown (Khalfin, 1984, 1985, 1986, 1987*a*, 1988), on the basis of probability theory, that the chaotic inflationary scenario is not realistic (and in many essential points this scenario is false). In this work I give corresponding proofs in the most general case, which is more general than in Khalfin (1984, 1985, 1986, 1987*a*, 1988) and Guth (1981). The discussion of the idea of a self-reproducing, chaotic inflationary universe, in connection with these proofs, will be given in a separate work.

2. Rubakov *et al.* (1982), Hawking (1985), Starobinsky (1985*a*), and Lyth (1984, 1985) obtained some limitations on inflationary models of the universe.[5] These limitations are general and do not depend on the details of the mechanism of the inflationary models. These limitations are based on the usual assumption of the homogeneity of the scale factor (the exponential inflation is spatially homogeneous). On the base of these limitations I obtained (Khalfin, 1985, 1986, 1987*a*, 1988) additional limitations on the initial conditions of the inflationary models. These additional limitations follow from the admissible space inhomogeneity of the scale factor (admissible space inhomogeneity of the exponential inflation). Here I recall the main formulas of these additional limitations (Khalfin, 1985, 1986, 1987*a*, 1988).

For all inflationary models the necessary condition on the admissible inhomogeneity of the initial scalar field ($\varphi(x)$ must be fulfilled [the notations correspond to Khalfin (1986)]

$$(\nabla\varphi)^2 \leq V(\varphi) \Rightarrow |\nabla\varphi| = \beta V^{1/2}(\varphi(x)), \qquad \beta \leq 1 \tag{1}$$

The Hubble "constant"

$$H(x) = [(8\pi/3m_{\rm p}^2)\, V(\varphi(x))]^{1/2} \tag{2}$$

depends on the scalar field $\varphi(x)$, which depends on x, and for this reason the Hubble constant is spatial inhomogeneous. Let us denote $\varphi_0 = \varphi(x=0)$, where $x=0$ is the mean point of a space region with size $l \simeq 2H^{-1}$, which by exponential (quasiexponential) inflation turn into the visible universe with the size $\simeq 10^{28}$ cm. The scale factor R, defined by (Kofman *et al.*, 1985),

$$R \simeq \exp\left(4\pi \frac{\varphi_0^2}{nm_{\rm p}^2}\right) \tag{3}$$

[5]I thank Dr. D. H. Lyth for information on Lyth (1984, 1985).

where the potential $V(\varphi)$ is $V(\varphi)=\alpha_n\varphi^n$, is given by

$$R \simeq \exp\left(4\pi\frac{\varphi_0^2}{nm_\mathrm{p}^2}\right)=\exp(64N^2), \qquad N\geq 1 \tag{4}$$

On the basis of (4), we have

$$\varphi_0=4\left(\frac{n}{\pi}\right)^{1/2} m_\mathrm{p}N \tag{5}$$

The exponential amplification in the scale factor R and in the Hubble constant inhomogeneities of the initial scalar field $\varphi(x)$ are (Khalfin, 1985, 1986, 1987*a*, 1988)[6]

$$\begin{aligned}
\frac{\delta R}{R}&=\frac{R(x=0\pm l)-R(x=0)}{R(x=0)}\\
&=\exp\left[\pm\beta\cdot 16\left(\frac{3}{2n}\right)^{1/2}N+\beta^2\frac{6}{n}\right]-1\\
&\simeq \pm 16\beta N\left(\frac{3}{2n}\right)^{1/2}\\
\frac{\delta H}{H}&=\frac{H(x=0\pm l)-H(x=0)}{H(x=0)}\\
&\simeq \pm\beta\frac{1}{16N}\left(\frac{3}{2n^{-1}}\right)^{1/2}
\end{aligned} \tag{6}$$

Note that the scale inhomogeneity, which corresponds to the scale inhomogeneity of the scale factor $R(x)$ and the Hubble constant $H(x)$, is connected with the definite choice of the coordinate frame. This coordinate frame is chosen such that for $\varphi(x)=\mathrm{const}(x)$, $\forall x$, the metric is defined by [see Starobinsky (1983) for notations]

$$\begin{aligned}
ds^2=dt^2-[&e^{Ht}a_{\alpha\beta}(x)+b_{\alpha\beta}(x)\\
&+e^{-Ht}c_{\alpha\beta}(x)+\cdots]\,dx^\alpha\,dx^\beta
\end{aligned} \tag{7}$$

Note that consideration is limited here to an "initial" inhomogeneity, and we do not consider "second" inhomogeneities, which are connected with the evolution of quantum fluctuations in the spatial inhomogeneity, exponentially inflated universe.

Let us now choose $N=1$. In this case the space region of size $l\simeq 2H^{-1}$ is transformed by the exponential inflation into the "visible" universe of size $\simeq 10^{28}$ cm. This minimally necessary exponential inflation gives us the

[6]Limitations analogous to (6) were independently obtained by Starobinsky (1985*b*).

solution[7] of some cosmological problems and at the same time gives us the possibility to connect the observable properties of the "visible" universe with the predictions of inflationary models for different N. In the case $N = 1$ formula (5) becomes

$$\varphi_0 = 4\left(\frac{n}{\pi}\right)^{1/2} m_p \tag{5a}$$

and formula (6) becomes

$$\frac{\delta R}{R} \simeq \pm 16\beta\left(\frac{3}{2n}\right)^{1/2}, \qquad \frac{\delta H}{H} \simeq \pm \tfrac{1}{16}\beta\left(\frac{3n}{2}\right)^{1/2} \tag{6a}$$

Experimental data (Abbott and Wise, 1984) indicate that the admissible space inhomogeneity of "our" universe has the value of order $\simeq 10^{-6}$. Then on the basis of (6a) we can give an estimate on the admissible inhomogeneity of the initial scalar field $\varphi(x)$ in the domain of size $l \simeq 2H^{-1}$:

$$\beta \lesssim 10^{-6} \tag{8}$$

This initial domain exponentially inflates to the "visible" universe. From (8) it follows that

$$|\nabla\varphi| \lesssim 10^{-6}\alpha_n^{1/2}\left[4\left(\frac{n}{\pi}\right)^{1/2} m_p\right]^{n/2} \tag{9}$$

For the potential $V(\varphi) = \frac{1}{4}\lambda\varphi^4$ [this potential was used in the chaotic scenario (Linde, 1983*a*,*b*)], to take into account bounds (Rubakov *et al.*, 1982; Hawking, 1985; Starobinsky, 1985*a*; Lyth, 1984, 1985) on λ we take from (9)

$$|\nabla\varphi| \lesssim 10^{-11} m_p^2 \tag{10}$$

For the potential[8] $V(\varphi) = \frac{1}{2}m^2\varphi^2$, to take into account bounds (Rubakov *et al.*, 1982; Hawking, 1985; Starobinsky, 1985*a*; Lyth, 1984, 1985) on m^2 we take from (9) the same (in the order of value) inequality as (10).

If we choose $N \gg 1$, which corresponds, according to (5), to a large value of the initial scalar field φ_0 [as was assumed in work on the chaotic inflationary scenario (Linde, 1984*a*, 1983*b*)], then the space in homogeneity (6) cannot be connected with possible observable data of the space inhomogeneity of the "visible" universe. This is evident because for $N \gg 1$ the space domain of size $l \simeq 2H^{-1}$ exponentially inflates to a universe whose size is much much greater than the size ($\simeq 10^{28}$ cm) of the "visible" universe. In this case a space domain of size smaller than $l \simeq 2H^{-1}$ inflates to the size of the "visible" universe and for this reason instead of (8) the limitations are much weaker until $\beta \simeq 1$. However, from our point of view the *a priori*

[7]For comments on this solution see the end of this report.

[8]A full investigation of the classical dynamics of this model is given in Belinsky *et al.* (1985).

choice $N \gg 1$ is nonphysical, first of all, because this choice does not follow necessarily from the solution of cosmological problems, and second (this is more essential) the predictions (6) following from this choice $N \gg 1$ on the space inhomogeneity of a universe whose size is much larger than the size of the "visible" ($\simeq 10^{28}$ cm) universe cannot be tested by observations (experiments) in principle. Predictions which cannot be tested by observations or by experiments are impossible to consider as scientific predictions. Usually, the work on the chaotic inflationary scenario (Linde, 1984*a*, 1983*b*) assumes that the choice $N \gg 1$ yields as its main prediction the homogeneity of "our" visible universe. But we knew, *a priori* to this scenario, that our visible universe is very homogeneous and from this point of view the conclusion from the chaotic scenario on the homogeneity of the visible (10^{28} cm) universe is not a prediction, but is no more than a "religious" explanation of this homogeneity, because the necessary choice $N \gg 1$, contrary to $N = 1$, cannot be proved by independent scientific reasons. The real physical prediction of the chaotic inflationary scenario with the choice $N \gg 1$ is the prediction of the great inhomogeneity of "our" universe on an "invisible" scale ($\gg 10^{28}$ cm) and this prediction is not a scientific prediction, because it cannot be tested by observations in principle.

3. The above bounds (9) and (10) are not the bounds on the parameters λ and m of the physical models as in Rubakov *et al.* (1982), Hawking (1985), Starobinsky (1985*a*), and Lyth (1984, 1985), bounds on the initial conditions [on the admissible inhomogeneity of the initial scalar field $\varphi(x)$]. These new bounds, which follow from the admissible space inhomogeneity of the "visible" universe, have no physical foundation and look like *ad hoc a posteriori* limitations, which are sufficient for the correctness of the inflationary model of the origin of "our" universe. Even if we understood the theory of the universe in the preinflation era (as might be possible in a future quantum theory of gravity) and could consider initial conditions of the inflationary era as consequences of the preinflationary era, we nevertheless could not solve the problem of the initial conditions. In fact, in this case the bounds (9), (10) will be simply transformed into the corresponding initial conditions of the preinflationary era. Only one radical way is possible for the solution of the initial conditions problem—namely, if the admissible inflation takes place for "almost all" initial conditions (compare this with the situation in the ergodic theorem, or with the existence of limited cycles for the solutions of nonlinear differential equations). In fact, this hope was the initial point in the foundation of the chaotic inflationary scenario as proposed by Linde (1983*a*,*b*). In this chaotic scenario initial scalar fields $\varphi(x)$ were assumed to be deterministic realizations of random (stochastic) fields. The problem of whether the deterministic realizations

(initial scalar fields) of random fields satisfy the necessary bounds and with what probability (what measure) is one of the classical problems of probability theory. This problem has both brilliant results and at the same time unsolved questions (see, for example, News, 1978). A trivial result states that the stationary random Gaussian process of "white noise" does not have continuous deterministic realizations. As an example of a very nontrivial result (Sudakov, 1976), I point out that for a sufficiently large set of the random fields the property of having continuous and bounded deterministic realizations satisfies the law "zero or identity." This means that either all deterministic realizations of these random fields have such a property or they do not. The definition of random fields corresponds to the definition of the probabilistic measure in the corresponding functional space. If the set of random fields (the set of probabilistic measures), deterministic realizations of which satisfy the necessary (see Section 2) conditions for the admissible inflation are sufficiently large, then the chaotic inflation scenario can in fact claim the scientific solution of many cosmological and fundamental conceptual problems. But if the deterministic realizations necessary for the admissible inflation have only degenerate, exotic[9] random fields or the measure of such necessary deterministic realizations is anomalously small, then the claims of the chaotic inflation scenario of the universe are false and cannot be considered as scientifically substantial (see the corresponding discussion in Section 1). In this work and in Khalfin (1985, 1986, 1987*a*, 1988) it was proved that the last alternative, unfortunately, is true for the chaotic inflationary scenarios of the universe.

The fact that the main problem for the chaotic inflationary models is that of estimating the probability (the measure) of the existence of deterministic realizations [initial scalar fields $\varphi(x)$] which have the necessary properties for the inflation was pointed out in Zel'dovich *et al.* (1987), Belinsky and Khalatnikov (1987), Starobinsky (1987), Page (1987*a,b*), Gibbons *et al.* (1987), and Hawking and Page (1987). As Belinsky notes (see Belinsky and Khalatnikov, 1987), the corresponding measure was defined in a voluntaristic way as uniform, following the "indifference principle." In the modern probability theory the danger of this principle is well known. Gibbons *et al.* (1987) defined, but only for homogeneous cosmological models, the "privileged" measure. As demonstrated in Page (1987*b*) and Hawking and Page (1987), using this measure reduces the ambiguity in the estimation of the measure of the deterministic realizations necessary for the properties of inflationary models.

What must be the *a priori* limitations on the random fields whose deterministic realizations are the initial scalar fields $\varphi(x)$ in the inflationary

[9]It is obvious that such a degenerate (exotic) random field is the deterministic field constructed "by hand" *ad hoc* as necessary for the inflation properties.

models? The main and fundamental limitation is connected with the existence of the horizon. From the probability theory point of view this means that the correlation function $r(x)$ of the homogeneous scalar field $\xi(x)$ [realizations of $\xi(x)$ are the initial scalar fields $\varphi(x)$]

$$r(x) = E[\xi(x) \cdot \xi^*(0)] \tag{11}$$

must have finite support of x:

$$r(x) = 0, \qquad |x| \geq l \tag{12}$$

where l is the size of the horizon. The condition (12) is very essential, because this condition reduces to the fundamental limitations on the spectral density $f(\lambda)$ of the random field $\xi(x)$. This spectral density $f(\lambda)$ is connected with the correlation function $r(x)$ on the basis of the Bochner–Kninchin theorem (Bochner, 1933) by the Fourier transform[10]

$$r(x) = \int_{-\infty}^{\infty} \exp(i\lambda x) \cdot f(\lambda)\, d\lambda \tag{13}$$

From (13) it follows that

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\lambda x) \cdot r(x)\, dx \tag{14}$$

Taking into account the property of the horizon (12), we have

$$f(\lambda) = \frac{1}{2\pi} \int_{-l}^{+l} \exp(-i\lambda x) \cdot r(x)\, dx \tag{14a}$$

From (14) it follows immediately that (Paley and Wiener, 1934) $f(\lambda)$ [on the basis of the horizon property (12)] is an entire function of λ and must necessarily satisfy the condition:

$$\left| \int_{-\infty}^{\infty} \frac{\log |f(\lambda)|}{1+\lambda^2}\, d\lambda \right| < +\infty \tag{15}$$

From (15) and from the fact that $f(\lambda)$ is an entire function it follows that: (a) $f(\lambda)$ cannot be a function of λ with finite support: $f(\lambda) \neq 0$ for $|\lambda| \geq \Lambda < +\infty$; (b) $f(\lambda)$ cannot be zero for finite intervals of $\lambda: f(\lambda) \neq 0$, $\lambda \in [\lambda_1, \lambda_2]$, $-\infty < \lambda_1 < \lambda_2 < \infty$; and (c) $f(\lambda)$ cannot decrease very rapidly (exponentially) for $\lambda \to \infty$,

$$|f(\lambda)| \geq A \exp(-\gamma |\lambda|^q), \qquad A > 0, \quad \gamma > 0, \quad q < 1; \qquad \lambda \to \infty \tag{16}$$

The properties (a)–(c) are very essential, but they were not taken into account in Linde (1986). From this point of view the main qualitative (and quantitative) statements of Linde (1986) are false. A discussion of that work in connection with this comment will be published separately.

[10]The condition (13) is true for random fields with absolutely continuous spectrum.

The definition of random fields $\xi(x)$, the deterministic realizations of which are the initial scalar fields $\varphi(x)$, originates with the quantum theory of these scalar fields. From this point of view the consideration below of the Gaussian random fields is natural. In addition, the limitation to Gaussian fields is also stipulated by purely mathematical reasons, because for non-Gaussian random fields we have no possibility to receive sufficiently meaningful estimates for interesting probabilities.

In the original work proposing the chaotic inflationary scenario (Linde, 1983*a,b*), it was explicitly assumed that deterministic realizations [scalar fields $\varphi(x)$] of the random fields exist which are *constant* in the domains of the size $l \simeq 2H^{-1}$. In fact, Linde (1983*b*, p. 178) states, "in the open (infinite) universe at $t \simeq t_{\rm p}$ there should exist infinitely many locally homogeneous and isotropic domains of size $l \gg m_{\rm p}^{-1}$, containing a *locally homogeneous* field φ such that $m_{\rm p} \lesssim \varphi \lesssim m_{\rm p}/\lambda^{1/4}$" [italics added]. And, "let us consider the evolution of the *locally homogeneous* field in the early universe. The part of the universe inside a domain filled with a *homogeneous* field φ expands as de Sitter space with the scale factor $a(t) = a_0 \exp(Ht)$, where $H = t\frac{8}{3}\pi V(\varphi)/m_{\rm p}^2]^{1/2}$" [italics added]. Linde (1983*a*) stated that the probability of such locally homogeneous scalar fields inside domains of size $l \simeq 2H^{-1}$ in the volume is nonzero. What is more, Linde (1983*a*) gives as a reference for the proof of this statement Linde (1983*c*), which in fact does not exist, nor does the corresponding proof exist in any of Linde's published work. But this proof cannot be in Linde's publications, because nowhere in these works is there a mathematical definition of the corresponding random fields, i.e., a mathematical definition of the measure in the corresponding functional space. In reality, for a very sufficiently large set of random fields the probability of the existence of locally homogeneous deterministic realizations is rigorously zero. In order not to complicate the mathematical proof with technical details, I limit myself here to one-dimensional random fields (processes).

The proof of the statement is based on the inequality (Lukacs, 1960)

$$|r(x)| < \left(1 - x^2 \frac{1-B^2}{8X^2}\right), \qquad x < X < \infty \tag{17}$$

where B and X are defined by

$$|r(x)| \leq B \leq 1 \qquad \text{for} \quad x \geq X < \infty \tag{18}$$

The existence of locally homogeneous deterministic realizations in the finite interval of size l means that $r(x)$ has the property

$$r(x) = 1, \qquad x \in [0, l] \tag{19}$$

It is obvious that [if $f(\lambda)$ is not a Dirac δ-function, which was assumed from the beginning] X exists such that $l < X < \infty$, and $B < 1$. Then the assumption on the existence of locally homogeneous deterministic realizations in the interval of size l [see (19)] evidently contradicts the inequality (17) and thus our statement is proven. The analogous proof is true in the many-dimensional case and for the general geometry (but not necessarily Euclidean geometry) of a space on which the initial scalar field $\varphi(x)$ was defined.

Recent publications on the chaotic inflationary scenario (see, for example, Linde, 1986) now refer to the almost homogeneous fields $\varphi(x)$. But estimates of the probability of the existence of almost homogeneous deterministic realizations of the random fields reduces to considering the reality of the idea of the chaotic inflationary scenario as proposed in Linde (1983*a,b*). Estimates of the probability of the existence of almost homogeneous deterministic realizations cannot be obtained beyond probability theory, which gives the necessary estimates. These estimates are obtained in the new mathematical results in the theory of random fields proved by B. S. Tzirel'son (unpublished, April 1985) (and in the simple case also by M. A. Lifshitz) and was stimulated by problems in Khalfin (1985). Taking into account recent work on the chaotic inflationary scenario in which the case of high-dimensional spaces was considered, here, in contrast to Khalfin (1985), I treat the results of Lifshitz and Tzirel'son (1986) in the most general case. Let $\xi(x)$, $x \in R^n$, be the homogeneous Gaussian random field with continuous[11] deterministic realizations, and $f(\lambda)$, $\lambda \in R^n$, be the spectral density of the absolutely continuous component of the spectral measure. Tzirel'son's theorem gives the exponential estimates (from above) for the probability of small deviations of deterministic realizations of the random fields near a fixed surface:

Theorem (B. S. Tzirel'son, 1985). Let the function A: $R^n \to R^1$ be continuous, and $B(x)$: $R^n \to [0, +\infty)$ be lower semicontinuous. Then

$$P\{\forall x \in R^n, |\xi(x) - A(x)| \leq B(x)\} \leq \inf_{K>0} \exp\left\{-K^n \int_{R^n} \eta[m^{-1/2}(K)B(x)]\, dx\right\} \tag{20}$$

where

$$\eta[c] = -\ln \int_{-c}^{+c} (2\pi)^{-1/2} \exp(-u^2/2)\, du$$
$$m(K) = \operatorname*{ess\,inf}_{\lambda \in R^n} \left[(2\pi K^{-1})^n \sum_{z \in \mathbb{Z}^n} f(\lambda + 2\pi K^{-1} z)\right] \tag{21}$$

From this theorem we get the following result.

[11] The scalar fields $\varphi(x)$, which are considered as the deterministic realizations of the random fields, must be not only continuous, but also differentiable in the chaotic inflationary scenario.

Corollary. Let $f(\lambda) \geq a|\lambda|^{-n-\alpha}$ for $|\lambda| \geq \Lambda$. For $\varepsilon > 0$ let

$$K = 2\pi\, e^{1/n}(\sqrt{M}/2)^{n/\alpha+1}\left(\frac{2}{\pi}\frac{\varepsilon^2}{a}\right)^{1/\alpha}$$

where $M = \max(n, 4)$. Let us assume that $K\Lambda \leq \pi$. Then, for every closed set $T \supset R^n$ and for every continuous function A: $T \to R$ the following inequality holds:

$$P\{\forall x \in T, |\xi(x) - A(x)| \leq \varepsilon\} \leq \exp[-\alpha(2\pi)^{-1}K^{-n} \text{ mes } T] \tag{22}$$

The proof of this Corollary essentially depends on the following geometric lemma by Tzirel'son.

Lemma (Tzirel'son, 1985). Let positive numbers d and b be such that $2d \leq b$. Then for every $\lambda \in R^n$ there exists $z \in \mathbb{Z}^n$ such that

$$d \leq |\lambda + bz| \leq \tfrac{1}{2}b\sqrt{M} \tag{23}$$

where $M = \max(n, 4)$.

Note that, based on consequence (c) of the property of the horizon (12) in the definition of the function $m(K)$, the summation takes place essentially on the infinite number of integer points of $\mathbb{Z}^n$.

Now we formulate Tzirel'son's Theorem and Corollary for the usual 3-dimensional case and flat initial surface. So let $n = 3$, $\Delta(x) = \varphi_0 = \text{const}(x)$, and $B(x) = \Delta = \text{const}(x)$, let $T = S^3$ be the sphere of the radius $l \simeq 2H^{-1}$, and let $\varepsilon = \Delta = \text{const}(x)$ and $\alpha = 3 + \delta$.

Theorem. Let $\xi(x)$, $x \in R^3$, be a random Gaussian homogeneous field with differentiable realizations $\varphi(x)$ and $f(\lambda)$, where $\lambda \in R^3$ is the spectral density of this field. Suppose that for all $K > 0$ there is defined a function $m(K)$:

$$m(K) = \operatorname*{ess\,inf}_{\lambda \in R^3}\left[\left(\frac{2\pi}{K}\right)^3 \sum_{z \in \mathbb{Z}^3} f\left(\lambda + \frac{2\pi}{K}z\right)\right] \tag{21a}$$

Then for all $x \in S^3$ the following exponential estimate holds:

$$P\{\forall x \in S^3, |\xi(x) - \varphi_0| \leq \Delta\} \leq \inf_{K>0} \exp\left\{-\frac{1}{K^3}\tfrac{4}{3}\pi l^3 \eta\left[\frac{\Delta}{m^{1/2}(K)}\right]\right\} \tag{20a}$$

where

$$\eta\left[\frac{\Delta}{m^{1/2}(K)}\right] = -\ln \int_{-\Delta/m^{1/2}(K)}^{+\Delta/m^{1/2}(K)} (2\pi)^{-1/2} \exp\left(-\frac{u^2}{2}\right) du$$

Corollary. Let $\xi(x)$, φ_0, and Δ be as in the theorem. Let $G>0$ and $Q>0$ be such that[12]

$$f(\lambda)=\frac{a}{|\lambda|^{5+\varepsilon}} \qquad \text{for} \quad |\lambda|\geq Q \tag{24}$$

The condition (24) guarantees differentiability of the scalar field $\varphi(x)$—the realizations of the homogeneous random Gaussian field $\xi(x)$. For $\Delta\geq 0$ we define the quantity q:

$$q=2\pi\, e^{1/3}(2\Delta^2/\pi a)^{1/2+\varepsilon}$$

Using the definition of the spectral density $f(\lambda)$ in terms of the Fourier transform of the correlation function of the random field $\xi(x)$, we can readily show that q has the dimensions of a length. We now suppose that $Qq\leq\pi$. Then for all $x\in R^3$ we have the following exponential estimate:

$$P\{\forall x\in S^3, |\xi(x)-\varphi_0|\leq\Delta\}\leq \exp\left(-\frac{2+\varepsilon}{6}\tfrac{4}{3}\pi\frac{l^3}{q^3}\right)\simeq \exp\left(-\tfrac{4}{9}\pi\frac{l^3}{q^3}\right) \tag{25}$$

The estimates (20a) and (25) give estimates of the probability of realizations with given inhomogeneity Δ in the sphere S^3 of the Gaussian random field. These estimates depend explicitly on a functional $m(K)$ of the spectral density of the Gaussian random field and thus are not universal. In any case, we must satisfy necessary fundamental limitations like (16), which follow from the existence of the finite horizon, and these limitations are satisfied in the corollary. Recently for the one-dimensional case $n=1$ Tzirel'son proved the universal estimate which directly follows from the existence of a finite horizon:

Theorem (Tzirel'son, 1988). Let $\{\xi(t)\}_{t\in(-\infty,\infty)}$ be a stationary (homogeneous) random Gaussian process with covariance function $\mathrm{Cov}(\xi(t_1)\xi(t_2))$ with finite support (finite horizon):

$$\mathrm{Cov}(\xi(t_1)\xi(t_2))=0 \qquad \text{for} \quad |t_1-t_2|>T_0<+\infty \tag{26}$$

Then for all $t\in(0, T_0)$ for $\varepsilon\to 0$ the following estimate holds:

$$\ln P\left\{\max_{S\in[0,t]}|\xi(S)|\leq\varepsilon\right\}\leq -(1+o(1))\frac{t/(T_0+t)}{4\alpha t/(T_0+t)}\left(\ln\frac{1}{\varepsilon}\right) \tag{27}$$

where

$$\alpha(x)\stackrel{\text{def}}{=}\frac{1}{\pi}\int_0^{\pi x}\ln \operatorname{ctg}\frac{u}{2}\,du \qquad \text{for} \quad x\in[0,1]$$

[12]The assumption (24) is admissible with the fundamental necessary bound (16).

For $x \to 0$, $\alpha(x) \simeq x \ln x$. The estimate (27) is universal, and depends only on the size of the horizon T_0 and does not depend on other details of the correlation (covariance) function. For higher dimensions ($n \geq 2$) the analogous general results are not known.

It follows from (25) that for $\Delta = 0$ the corresponding probability of realizations that are constant (homogeneous) in the sphere S^3 is strictly equal to zero. This result was proved above in the general case and not only for Gaussian random fields. The estimates (20a) and (25) are, of course, of greatest interest for $\Delta \neq 0$. For sufficiently small Δ, the estimates (20a) and (25) are sufficiently accurate. We now recall the estimates obtained earlier for the permitted inhomogeneity of the initial scalar fields, namely (8). Then

$$\Delta \simeq 10^{-6} \varphi_0 \tag{28}$$

As follows from (24), a is proportional to the variance $\bar{\varphi}_0^2$. In the chaotic scenario, it is assumed that $\varphi_0/(\bar{\varphi}_0^2)^{1/2} \simeq 1$. Then, from (25), on the basis of (28), taking into account that $l \simeq 2H^{-1}$, we obtain the final exponential estimate:

$$P\{\forall x \in S^3, |\xi(x) - \varphi_0| \leq 10^{-6}\varphi_0\} \leq \exp(-10^{34}) \quad (!) \tag{29}$$

The estimate (25) corresponds to $N = 1$—the case which is natural and from the physical point of view is justified. For $N \gg 1$ the estimates corresponding to (29) are less limited, but as was argued above, the case $N \gg 1$ is non-physical and has no scientific foundation. I point out once more that prior to the result of Lifshitz and Tzirel'son (1986) there were no mathematical estimates of the probability of the admissible space inhomogeneity of the initial scalar field.

From (29) it follows that the probability of the origin of "our" universe (and "all of us") with the admissible space inhomogeneity is fantastically small $\exp(-10^{34})$, but the probability of the origin of universes that are not "ours" is fantastically larger than the probability of the origin of "our" universe.

As we know from quantum theory, especially after the fundamental inequality of Bell (1965) (see also Cirelson, 1980; Khalfin and Tzirel'son, 1985), "God plays dice," but it is hard to represent that with such fantastic accuracy.

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